Verified Solution Methods for Markov Decision Processes

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Abstract

We formally verify executable algorithms for solving Markov decision processes (MDPs) in the interactive theorem prover Isabelle/HOL. Then, we verify executable dynamic programming algorithms and approximate algorithms using linear programming certification. Currently, we are building verified certificate checkers for quantitative model checking.

Motivation

- **Markov decision processes (MDPs)** model systems where an agent maximizes rewards under uncertainty
- Application areas are often **safety-critical** and real-world scenarios: planning, reinforcement learning, model checking, operations research

⇒ Correct and robust software is important
⇒ We use **interactive theorem provers** to develop trustworthy software for MDPs

Isabelle/HOL

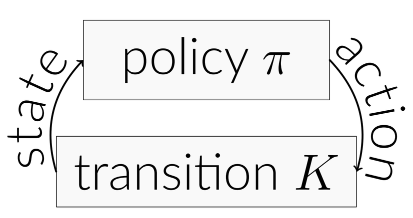
- We use **Isabelle/HOL** for our developments
- Isabelle/HOL is an interactive theorem prover
- Math library + AFP: probability theory, limits, etc.
- Powerful automation: *sledgehammer* and *auto*

- **Code extraction** ⇒ verified executables
- **Highly trustworthy**: high-level proofs ⇒ axioms

Markov Decision Processes

An MDP consists of

- sets of **states** S and **actions** A_s for each state
- **transition probabilities** $K : S \times A \to P(S)$
- **rewards** $r : S \times A \to \mathbb{R}$
- **discount factor** $\lambda \leq 1$



- **Goal**: maximize reward by optimizing a policy
- A **policy** selects an action based on the states and actions observed

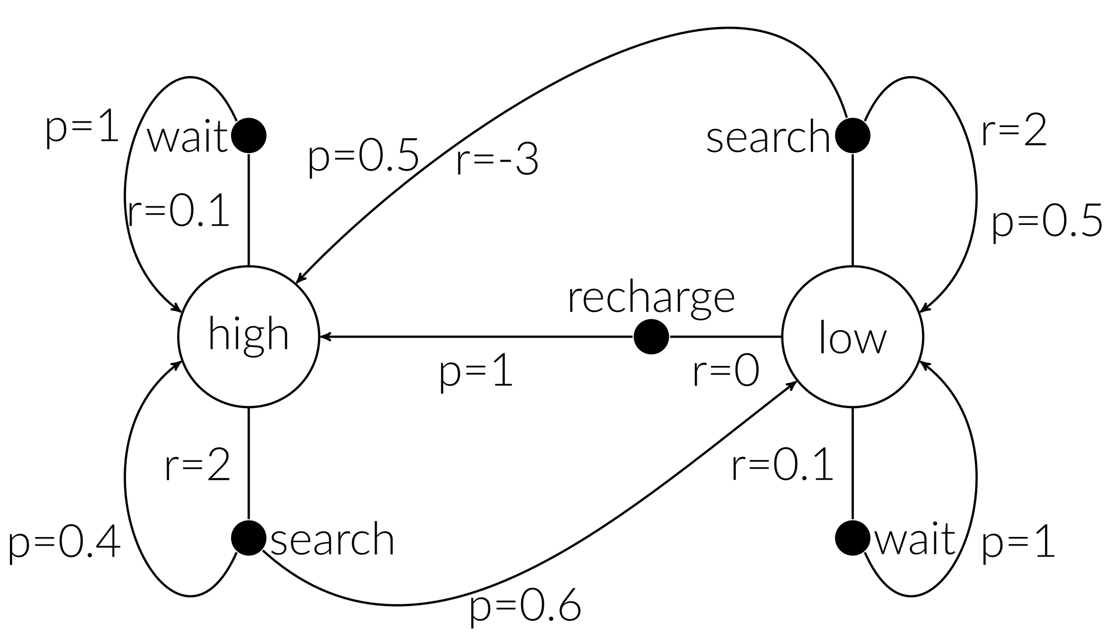


Figure 1. The robot can choose between searching for treasure, waiting, or recharging. States indicate battery levels.

Expected Total Reward

- Expected total discounted reward ν_π is the cumulative reward collected during a run:

$$\nu_\pi(s) = \lim_{n\to\infty} \mathbb{E}_{\omega\sim T_\pi(s)} \left[\sum_{i<n} \lambda^i \cdot r(\omega_i)\right]$$

- The discount factor makes the agent **short-sighted**

Dynamic Programming

- The **Bellman optimality operator**

$$\mathcal{L}(v) = \sup_\pi r_\pi + \lambda \mathcal{P}_\pi v$$

converges to the **optimal value** ν^*:

$$\lim_{i\to\infty} \mathcal{L}^i(v) = \nu^*.$$

- We formalize
  - (Gauss-Seidel) Value iteration
  - (Modified) Policy Iteration
- Refine algorithms to efficiently executable code
- Faster:
  - **begin with value iteration on floats**,
  - then one iteration with precise arithmetic

Value Iteration in Isabelle/HOL

```
function value_iteration ::
  ‹real ⇒ ('s ⇒b real) ⇒ ('s ⇒b real)›
where
  "value_iteration eps v = (
  if 2 * l * dist v (ℒb v) < eps * (1-l) ∨ eps ≤ 0
  then ℒb v
  else value_iteration eps (ℒb v))"
```

Benchmarks

- Benchmark problems:
  International Planning Competition 2018
- Precise arithmetic is expensive, certification is much more competitive

		VI			GS			Certif.	
			Verified			Verified			
Domain	Instances	Prism	$\mathbb{R}$	$\mathbb{F}$	Prism	$\mathbb{R}$	$\mathbb{F}$	VI	GS
traffic	4	4	4	4	4	2	4	4	4
elevators	8	5	2	6	5	1	6	6	6
game-of-life	3	3	–	3	3	–	3	3	–
manufacturer	2	2	–	2	2	–	2	2	2
luck	5	5	5	5	5	5	5	5	5
skill-teaching	8	6	4	7	6	3	7	6	6
tireworld	6	4	4	4	4	4	4	4	4
wildlife	8	6	5	8	6	4	8	8	8

Table 1. Table with number of instances solved by different algorithms. Columns 2,3 show the performance of PRISM vs. our implementations: $\mathbb{R}, \mathbb{F}$ indicate precise or floating-point arithmetic. Column 4 displays the performance of our certification approach.

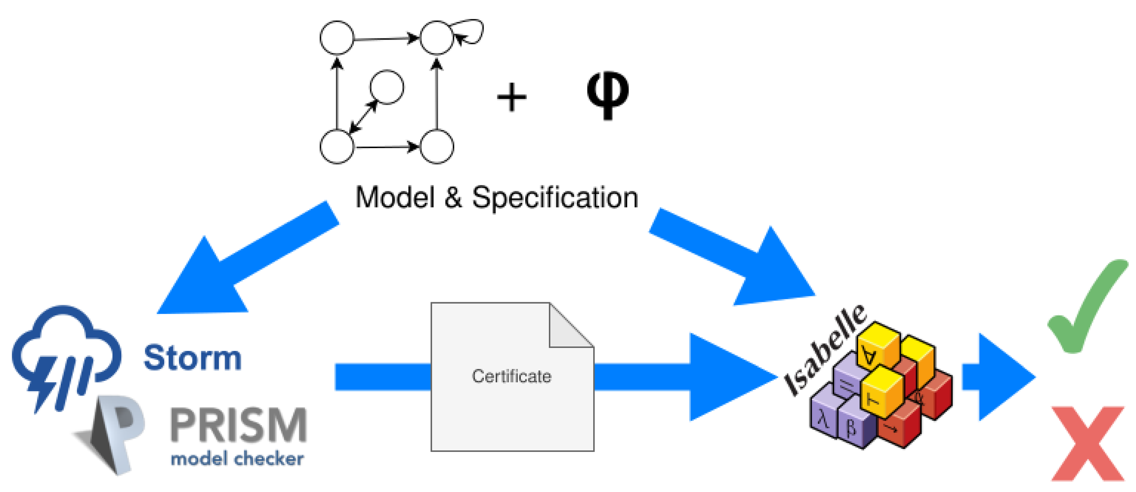
Example Proof

```
lemma contraction_L: "dist (L p v) (L p u) ≤ l * dist v u"
proof -
  have aux: "L p v s - L p u s ≤ l * dist v u"
    if lea: "(L p v s) ≥ (L p u s)" for v s u
  proof -
    have "L p v s - L p u s = l *R (𝒫₁ p v - 𝒫₁ p u) s"
      by (simp add: L_def algebra_simps)
    also have "... ≤ l * ¦𝒫₁ p (v - u) s¦"
      by (auto simp: blinfun.diff_right)
    also have "... ≤ l * norm (𝒫₁ p (v - u))"
      using abs_le_norm_bfun by fastforce
    also have "... ≤ l * dist v u"
      by (auto simp: 𝒫₁.rep_eq  dist_norm)
    finally show ?thesis
      by auto
  qed
  have "dist (L p v s) (L p u s) ≤ l * dist v u" for s
    using aux aux[of v _ u] by (cases "L p v s ≥ L p u s")
      (auto simp: dist_real_def dist_commute)
  thus "dist (L p v) (L p u) ≤ l * dist v u"
    by (simp add: dist_bound)
qed
```

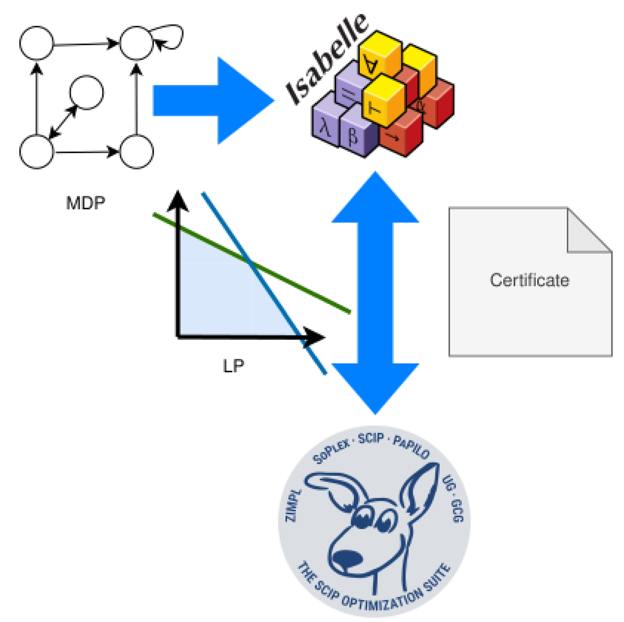
Conclusion

- ITPs are suitable for developing executable algorithms for reasoning under uncertainty
- Powerful proof automation + a large library of formal proofs were essential
- Certification is key for large state spaces
- Future work: Floating-point guarantees, Monte-Carlo algorithms

Quantitative Model Checking



Linear Programming Certification



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[2] Maximilian Schäffeler and Mohammad Abdulaziz. Formally verified solution methods for markov decision processes. In *AAAI 2023*.

[3] Maximilian Schäffeler and Mohammad Abdulaziz. Verified algorithms for solving markov decision processes. *Archive of Formal Proofs*, December 2021. `https://isa-afp.org/entries/MDP-Algorithms.html`, Formal proof development.

Funding This work was supported by the Research Training Group GRK 2428 CONVEY of the German Research Council (DFG)